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Gravitational contribution to fermion masses

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Abstract. In the context of a non-linear gauge theory of the Poincaré group, we show that covariant derivatives of Dirac fields include a coupling to the translational connections, manifesting itself in the matter action as a universal background mass contribution to fermions.

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1 Introduction

Conceived as an alternative to the standard general relativistic metric approach to gravity, gauge theories of spacetime groups describe gravitational forces in close analogy to the remaining interactions [1–9]. The Lorentz group and the $GL(4\,,R)$ group are usual candidates proposed by various authors [2,3,10,11] to play the role of local symmetries. Instead, Hehl et al. [4,8,12,13] consider gravity as the gauge theory of either the Poincaré or the affine group: in any case of a group including translations. Actually, the interpretation of tetrads as a certain kind of translational connections allows for an uniform description of all known interactions, gravity included, in terms of gauge potentials declared as the unique force mediators [14–17].

We are interested in analyzing the consequences for matter fields of considering translations included in the gauge group, as for instance in the Poincaré gauge theory (PGT) of gravity, where the full Poincaré group is treated as the local gauge group of a Yang–Mills type theory. Given that such approach is a serious candidate to become the fundamental theory of gravity, obviously we must know how the corresponding Poincaré covariant derivatives of matter fields look like, with both the homogeneous Lorentz group contributions and those of translations taken into account. The present paper is devoted to give an answer to the question how translational connections couple in particular to Dirac fields.

Independently of the interest of PGT in itself, the fact that we choose it with preference to a more general gauge theory of gravitation, such as metric-affine gravity (MAG), is partly determined by a technical reason, namely the possibility of building an explicit matrix representation of the Poincaré algebra. In fact, besides the usual spin operators $\sigma_{\alpha\beta}$ constituting the representation of the Lorentz generators acting on Dirac fields, one can introduce the complementary realization π_μ of the translational generators. The affine group is more problematic to deal with due to the fact that no finite-dimensional spinor representation of $GL(4\,,R)$ exists [8].

As an unexpected consequence of the explicit construction of covariant Poincaré derivatives with intrinsic translations, we find that the translational connections contribute to the Dirac action with a fermion mass term of PGT-gravitational nature. Such a result is exclusive for a certain kind of gauge theories of gravity, having nothing to do either with ordinary general relativity or with gauge approaches based on spacetime groups not including translations. More precisely, we derive the background fermion mass from the non-linear approach to PGT established by us in a number of previous papers [14–18]. There we developed a suitable treatment of spacetime groups with translations, explaining the identification of tetrads as (non-linear) translational connections, and one of us proposed an adapted fiber bundle description [19].

In next section we review a few main concepts, necessary to deduce the key formula (26) expressing non-linear connections in terms of linear ones. Then in Sect. 3 we apply the non-linear procedure to the Poincaré group, paying special attention to covariant derivatives of Dirac fields, and finally in Sect. 4 we build the matter action showing the emergence of the translation-induced mass term.

2 Generalized bundle structure of gauge theories

2.1 Composite fiber bundles

The ordinary gauge theory of a given Lie group G is known to have the structure of a principal bundle $P(M\,,G)$

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equipped with a connection, being matter fields defined on associated bundles. However, gauge theories involving non-linearly realized local symmetries, as for instance gauge theories of spacetime groups, require a slight modification of this bundle scheme, as discussed in [19]. The composite fiber bundles studied there are particularly suitable to highlight the underlying geometry of gauge theories of groups including translations, such as the Poincaré gauge theory of gravity, thus constituting the main support of the present paper. Let us briefly remind the reader of its defining features. For what follows, see [20–23], as well as [24], pp. 54 and 57.

Let $\pi_{PM} : P \rightarrow M$ be a principal fiber bundle with structure Lie group G, and let H be a closed subgroup of G. The quotient space G/H constitutes a manifold on which G acts on the left in a natural way. Then it is possible to build the P-associated bundle $\pi_{\Sigma M} : \Sigma \rightarrow M$ with standard fiber G/H and with total space consisting of the quotient space $\Sigma = (P \times G/H)/G$ of the Cartesian product $P \times G/H$ by the right action of G defined as $P \times G/H \ni (u\,, \xi\,) \rightarrow (ug\,, g^{-1}\xi\,) \in P \times G/H\,, g \in G$. The total space Σ can be identified with the quotient space P/H of P by the right action of H on P, and consequently one finds $P(\Sigma\,, H)$ to be a principal fiber bundle with structure group H and with well defined projection $\pi_{P\Sigma} : P \rightarrow \Sigma$ onto the base space $\Sigma = P/H$; see Proposition 5.5 of [24]. Indeed, each orbit uH through $u \in P$ – diffeomorphic to the standard fiber H – projects into a single element (a left coset) of P/H.

Non-linearly realized gauge theories to be studied here differ from ordinary gauge theories in that they are based on principal bundles $P(M\,, G)$ whose structure group G is reducible to a closed subgroup H. According to Proposition 5.6 of [24], such a reducibility of the structure group G to $H \subset G$ is guaranteed if and only if a cross section $s_{M\Sigma} : M \rightarrow \Sigma = P/H$ of the associated bundle Σ exists. Furthermore, there is a one to one correspondence between sections $s_{M\Sigma}$ and the reduced subbundles of $\pi_{P\Sigma} : P \rightarrow \Sigma$ consisting of the set of points $u \in P$ such that

$$\pi_{P\Sigma}(u) = s_{M\Sigma} \circ \pi_{PM}(u)\,; \tag{1}$$

see [24]. From condition (1) follows trivially

$$\pi_{PM} = \pi_{\Sigma M} \circ \pi_{P\Sigma}\,, \tag{2}$$

providing a decomposition of the total projection π_{PM} into partial projections. Accordingly, the principal bundle $\pi_{PM} : P \rightarrow M$ transforms into the composite bundle

$$\pi_{\Sigma M} \circ \pi_{P\Sigma} : P \rightarrow \Sigma \rightarrow M\,. \tag{3}$$

In (3) we distinguish two bundle sectors, characterized respectively by the partial projections

$$\pi_{P\Sigma} : P \rightarrow \Sigma\,, \quad \pi_{\Sigma M} : \Sigma \rightarrow M\,. \tag{4}$$

The latter one, with standard fiber G/H, can be seen as an intermediate space, in the sense that it is built over the primary base space M, and simultaneously plays a role as the base space of the principal bundle $\pi_{P\Sigma} : P \rightarrow \Sigma$ with structure group H. More precisely, in the context of composite bundles one can regard the G-diffeomorphic fibers of $P(M\,, G)$ as being, say, *bent* into two sectors, corresponding respectively to the fibers H of $\pi_{P\Sigma} : P \rightarrow \Sigma$ and G/H of $\pi_{\Sigma M} : \Sigma \rightarrow M$. The H-diffeomorphic fiber branches are attached to points of the *intermediate base space* Σ, which trivialize locally as $(x\,, \xi)$, with ξ coordinatizing the fiber branches G/H attached to $x \in M$.

In parallel to (2), the local sections $s_{MP} : M \rightarrow P$ are decomposed as

$$s_{MP} = s_{\Sigma P} \circ s_{M\Sigma}\,, \tag{5}$$

see Sect. V of [19]. In terms of suitable zero sections, denoting as σ_{MP} those corresponding to s_{MP}, and so on, the sections in (5) become respectively

$$s_{MP} = R_{\tilde{g}} \circ \sigma_{MP}\,, \quad \tilde{g} \in G\,, \tag{6}$$

$$s_{\Sigma P} = R_a \circ \sigma_{\Sigma P}\,, \quad a \in H\,, \tag{7}$$

and

$$s_{M\Sigma} = R_b \circ \sigma_{M\Sigma}\,, \quad b \in G/H\,. \tag{8}$$

The conditions

$$\tilde{g} = b \cdot a\,, \quad \sigma_{\Sigma P} \circ R_b = R_b \circ \sigma_{\Sigma P} \tag{9}$$

ensure that, in analogy to (5), the relation

$$\sigma_{MP} = \sigma_{\Sigma P} \circ \sigma_{M\Sigma} \tag{10}$$

also holds. The usefulness of this structure will become evident in the following.

In summary, the composite fiber bundles (3) provide the mathematical foundation for gauge theories involving non-linear gauge realizations (as the generalization of induced representations). Relevant physical theories comprised among the concerned ones are on the one hand the standard model – since a correspondence between non-linear realizations and spontaneous symmetry breaking exists [25] – and on the other hand non-linear gauge theories of gravity, as developed below. Non-linear realizations characteristic for such theories take place on principal fiber bundles $P(M\,, G)$ whose structure group G is reducible to a closed subgroup $H \subset G$. While the total symmetry remains that of the gauge group G, one exploits the possibility of working with the explicit symmetry H of the principal subbundle of $\pi_{P\Sigma} : P \rightarrow \Sigma$ whose base space is the total space of $\pi_{\Sigma M} : \Sigma \rightarrow M$. (The sections $s_{M\Sigma}$ defined on the latter bundle are identified as Goldstone fields [20].)

2.2 Non-linear realizations in composite bundles

In [19] it was shown that the composite bundle structure defined by (4)–(10) provides the natural framework to deal with non-linear gauge realizations, exactly as standard principal bundles constitute the arena for the ordinary gauge treatment of groups. Actually, the main results on non-linear realizations [26–31] are easily derived. So, the non-linear gauge transformation equation

$$L_g \circ \sigma_{\Sigma P}(x\,,\xi) = R_h \circ \sigma_{\Sigma P}(x\,,\xi') \tag{11}$$

is obtained by comparing two bundle elements, both with the form (7), differing from each other by the left action L_g of elements $g \in G$, the latter being local in the sense that $g = g(x)$, $x \in M$; see [19] for details. Regarding (11) referred to the base space M, it manifests itself as a vertical bundle automorphism not affecting $x \in M$, in analogy to ordinary gauge transformations. Nevertheless, when referred to the intermediate base space $\Sigma \simeq M \times G/H$, the action of L_g not only transforms the sections $\sigma_{\Sigma P}$ vertically along the H fiber branches by means of R_h, $h \in H$, but simultaneously it induces a transformation affecting the points $(x\,,\xi) \in \Sigma$, thus mapping H fiber branches into fiber branches defined on different Σ-points (as expected for spacetime groups, in particular for translations; see (33) below).

In order to deal with ordinary geometrical objects defined on the base manifold M, we pull back to the latter, by means of $s_{M\Sigma}$, the quantities defined on the plateau Σ. Taking into account the property of pullbacks when applied to functions φ, namely $f^*\varphi = \varphi \circ f$, we first define the pullback of $\sigma_{\Sigma P}$ as

$$\sigma_\xi(x) := (s^*_{M\Sigma}\sigma_{\Sigma P})(x) = \sigma_{\Sigma P} \circ s_{M\Sigma}(x)\,. \tag{12}$$

Then we calculate $s^*_{M\Sigma}(L_g \circ \sigma_{\Sigma P}) = L_g \circ \sigma_{\Sigma P} \circ s_{M\Sigma} = L_g \circ \sigma_\xi$ and $s^*_{M\Sigma}(R_h \circ \sigma'_{\Sigma P}) = R_h \circ \sigma'_{\Sigma P} \circ s_{M\Sigma} = R_h \circ \sigma_{\xi'}$, so that (11) gives rise to

$$L_g \circ \sigma_\xi(x) = R_h \circ \sigma_{\xi'}(x)\,. \tag{13}$$

In (13) (see (6.6) of [19]) one recognizes the fundamental equation for non-linear realizations [26,6,7].

The non-linear gauge transformations of fields induced by (13) are deduced in Sect. VIII of [19]. Taking in (13) $h \approx I + \mu$ to be infinitesimal, with μ defined on the Lie algebra of H, the fields $\psi(\sigma_\xi(x)) := (\sigma^*_\xi\,\psi)(x)$ of any given representation space of H are found to transform infinitesimally under G as

$$\delta\psi(\sigma_\xi(x)) := \sigma^*_{\xi'}\,\psi - (L_g \circ \sigma_\xi)^*\,\psi \approx \rho(\mu)\psi(\sigma_\xi(x))\,, \tag{14}$$

being $\rho(\mu)$ the suitable representation of the H-algebra element μ. (See (8.11) of [19].) Equations (13) and (14) summarize the main results of [26]. In the non-linear approach, the relevant fact is that the fields ψ of representation spaces of $H \subset G$ also constitute a representation space for the non-linear action (13) of the full group G.

2.3 Bundle approach to non-linear connections and covariant derivatives

Covariant derivatives of the fields in (14) require the introduction of suitable (non-linear) connections on M. As a crucial result for this purpose, in the present paragraph we will derive (26) below, implicit in [19] but not explicitly given there, expressing the non-linear connections in terms of standard (linear) gauge potentials.

Depending on the bundle base space we consider, that is, either M or the *plateau* Σ, at least two alternative expressions can be given for the Ehresmann connection form. On the one hand, taking the quantities in (6) into account,

$$\omega = \tilde{g}^{-1}(\,d + \pi^*_{PM}\,A_M)\,\tilde{g}\,, \tag{15}$$

involving the ordinary gauge potential A_M on the base space M, defined as the pullback

$$A_M = \sigma^*_{MP}\,\omega\,. \tag{16}$$

On the other hand, with (7) at view,

$$\omega = a^{-1}(\,d + \pi^*_{P\Sigma}\,\Gamma_\Sigma)\,a\,, \tag{17}$$

where we introduce the non-linear connection on the intermediate space Σ, turning out to be the pullback

$$\Gamma_\Sigma = \sigma^*_{\Sigma P}\,\omega\,. \tag{18}$$

Since $\tilde{g} = b{\cdot}a$, see (9), a comparison of (15) and (17) yields

$$\pi^*_{P\Sigma}\,\Gamma_\Sigma = b^{-1}(\,d + \pi^*_{PM}\,A_M)\,b\,. \tag{19}$$

From the defining condition $\pi_{P\Sigma} \circ \sigma_{\Sigma P} = id_\Sigma$ for sections follows $\sigma^*_{\Sigma P}\,\pi^*_{P\Sigma} = id_{T^*(\Sigma)}$, so that (19) gives rise to

$$\Gamma_\Sigma = \sigma^*_{\Sigma P}\,[\,b^{-1}(\,d + \pi^*_{PM}\,A_M)\,b\,]\,. \tag{20}$$

We operate on (20) taking into account that, in terms of the pulled back quantity

$$\hat{b}(x\,,\xi) := b \circ \sigma_{\Sigma P}(x\,,\xi)\,, \tag{21}$$

the relations $\sigma^*_{\Sigma P}\,(b^{-1}d\,b) = \hat{b}^{-1}d\,\hat{b}$, and $\sigma^*_{\Sigma P}\,R^*_b = R^*_{\hat{b}}\,\sigma^*_{\Sigma P}$ hold, in view of $b^{-1}\pi^*_{PM}\,A_M\,b = R^*_b\,\pi^*_{PM}\,A_M$ and in view of $\pi^*_{PM} = \pi^*_{P\Sigma}\,\pi^*_{\Sigma M}$ due to (2). We find

$$\Gamma_\Sigma = \hat{b}^{-1}(\,d + \pi^*_{\Sigma M}\,A_M)\,\hat{b}\,. \tag{22}$$

Pulling back (22), defined on Σ, by means of $s^*_{M\Sigma}$, compare with (12), we get

$$\Gamma_M = s^*_{M\Sigma}\,\Gamma_\Sigma \tag{23}$$

as the non-linear connection, defined on the base space M, which we will deal with in the following. Obviously, in view of (18) and (12)

$$\Gamma_M = s^*_{M\Sigma}\,\sigma^*_{\Sigma P}\,\omega = \sigma^*_\xi\,\omega\,. \tag{24}$$

Calculations analogous to those leading from (20) to (22) allow us to find, in terms of the new pulled back quantity

$$\tilde{b}(x) := b \circ \sigma_{\Sigma P} \circ s_{M\Sigma}(x) = b(\sigma_\xi(x))\,, \tag{25}$$

the relation

$$\Gamma_M = \tilde{b}^{-1}(d + A_M)\,\tilde{b}\,, \tag{26}$$

between the non-linear connection Γ_M and the linear connection A_M, that is, between the alternative pullbacks (24) and (16) of the connection 1-form ω to M. Our deduction of (26) provides a geometrical interpretation of (19) of [26], (6) of [25] and (2.7) of [32], while it shows the incompleteness of (2.15) of [28] or (22) of [30]. The importance of (26) for what follows becomes evident in view of the transformation properties of Γ_M, given in (7.14) of [19], namely

$$\delta\Gamma_M = \sigma^*_{\xi'}\,\omega - (L_g \circ \sigma_\xi)^*\,\omega \approx -(\,d\mu + [\Gamma_M\,,\mu])\,, \quad (27)$$

with μ being the same H-algebra element as in (14). Equation (27) shows that only the H-algebra components of Γ_M still transform inhomogenously as H-connections, while the G/H-algebra components transform as H-tensors. According to (14) and (27), the non-linear covariant differential defined as

$$D\psi := (d + \rho(\Gamma_M))\psi \quad (28)$$

transforms as an H-covariant differential

$$\delta D\psi = \rho(\mu)D\psi \quad (29)$$

under non-linear gauge transformations (13) of the full group G. The general procedure established here will be applied in the next section to G as the Poincaré group and H as the Lorentz group in order to derive PGT.

3 Non-linear Poincaré gauge theory of gravity

3.1 Poincaré covariant derivatives

The main results of the previous section are summarized in the transformation law (13) and the induced field transformation (14), plus the relation (26) between the non-linear connection (24) and the linear one (16), with the corresponding non-linear connection transformation (27). In terms of these elements, one defines the covariant differential (28) transforming as (29).

Now, in order to perform explicit calculations, we need to transform (13) into a more manageable formula. From (12) with (8), (9) and (10), we get $\sigma_\xi(x) = R_b \circ \sigma_{MP}(x)$. (In the latter equation we identify $b = \sigma^{-1}_{MP}(x)\cdot\sigma_\xi(x) = b(\sigma_\xi(x)) =: \tilde{b}(x)$ as given by (25).) Analogously, $\sigma_{\xi'}(x) = R_{b'} \circ \sigma_{MP}(x)$. Replacing these values into (13), it follows that $L_g \circ R_b \circ \sigma_{MP}(x) = R_h \circ R_{b'} \circ \sigma_{MP}(x)$. Finally, since $\sigma^{-1}_{MP}(x)\cdot g \cdot \sigma_{MP}(x) = g$, we find

$$g\cdot b = b'\cdot h\,. \quad (30)$$

Equation (30) is the form of (13) appearing in [26], appropriate for practical computational purposes, with b being (25) and thus identical with $\tilde{b}$ in (26).

Now we merely apply the general formalism mechanically to the gauge group G = Poincaré, with H = Lorentz. (Other choices of H have been studied elsewhere [16].) In (30) we replace the infinitesimal group elements $g \approx I + \mathrm{i}\,\epsilon^\mu P_\mu + \mathrm{i}\,\beta^{\alpha\beta}L_{\alpha\beta}$ of the Poincaré group and $h \approx I + \mathrm{i}\,\mu^{\alpha\beta}L_{\alpha\beta}$ of the homogeneous Lorentz group, and we parametrize b and b' respectively as $b = \mathrm{e}^{-\mathrm{i}\,\xi^\mu P_\mu}$ with finite translational parameters ξ^μ, and $b' = \mathrm{e}^{-\mathrm{i}\,\xi'^\mu P_\mu}$ with $\xi'^\mu \approx \xi^\mu + \delta\xi^\mu$. Then, taking into account the Poincaré commutation relations

$$[\,L_{\alpha\beta}\,, L_{\mu\nu}] = -\mathrm{i}\,(o_{\alpha[\mu}L_{\nu]\beta} - o_{\beta[\mu}L_{\nu]\alpha})\,, \quad (31)$$

$$[\,L_{\alpha\beta}\,, P_\mu] = \mathrm{i}\,o_{\mu[\alpha}P_{\beta]}\,, \quad [\,P_\mu\,, P_\nu] = 0\,, \quad (32)$$

with the help of the Hausdorff–Campbell formula, (30) yields on the one hand the value $\mu^{\alpha\beta} = \beta^{\alpha\beta}$ for the H-parameter, and on the other hand

$$\delta\xi^\mu = -\xi^\nu\beta_\nu{}^\mu - \epsilon^\mu\,. \quad (33)$$

Observe how the transformations (33) of the translational parameters resemble those of Cartesian coordinates.

Let us now pay attention to the connections. Starting with the ordinary linear ones for the Poincaré group, say

$$A_M = -\mathrm{i}\,\Gamma^{\alpha\beta}L_{\alpha\beta} - \mathrm{i}\,\overset{(T)}{\Gamma^\mu}P_\mu\,, \quad (34)$$

we make use of (26) to construct, in terms of (34) and of $b = \mathrm{e}^{-\mathrm{i}\,\xi^\mu P_\mu}$, the non-linear connections

$$\Gamma_M = -\mathrm{i}\,\Gamma^{\alpha\beta}L_{\alpha\beta} - \mathrm{i}\,\vartheta^\mu P_\mu\,, \quad (35)$$

where simple calculations yield for the non-linear translational connection the structure

$$\vartheta^\mu := D\xi^\mu + \overset{(T)}{\Gamma^\mu}\,, \quad (36)$$

where $D\xi^\mu := d\xi^\mu + \Gamma_\nu{}^\mu\xi^\nu$. More explicitly, since all quantities are pulled back to the base space M, (36) reads

$$\vartheta^\mu = dx^i\left(\partial_i\xi^\mu + \Gamma_{i\nu}{}^\mu\xi^\nu + \overset{(T)}{\Gamma^\mu_i}\right) =: dx^i e_i{}^\mu\,, \quad (37)$$

where we introduce the usual notation $e_i{}^\mu$ for *vierbeins* in order to show the identification we make of the non-linear translational connections with the tetrads. Such an interpretation of tetrads is possible since, in view of (27), they obey the gauge transformations

$$\delta\vartheta^\mu = -\vartheta^\nu\beta_\nu{}^\mu\,. \quad (38)$$

In addition we find for the Lorentz part of (35)

$$\delta\Gamma_\alpha{}^\beta = D\beta_\alpha{}^\beta\,. \quad (39)$$

As a consistence condition for (33), (36), (38) and (39) follows the transformation of the linear translational connection

$$\delta\overset{(T)}{\Gamma^\mu} = -\overset{(T)}{\Gamma^\nu}\beta_\nu{}^\mu + D\epsilon^\mu\,. \quad (40)$$

Comparison of (40) with the transformations (38) of the non-linear translational connections (36) clarify why the latter, as a result of the non-linear approach, can play the role of tetrads. Actually, tetrad variations (38) constitute a particular case of the above mentioned fact that the non-linear connection components associated to generators of G not belonging to H behave as H-tensors.

With the previous results at hand, the main task of the present paragraph is to construct the Poincaré covariant derivatives of matter fields. As shown by (14), the gauge action of the full Poincaré group G takes place through the representation $\rho(\mu) = \mathrm{i}\,\mu^{\alpha\beta}\rho(L_{\alpha\beta})$ of the algebra of the Lorentz group H, acting on fields of arbitrary representation spaces of H. In particular, for Dirac fields we take the spinor generators $\rho(L_{\alpha\beta}) = \sigma_{\alpha\beta}$ as given by (47) below, having $\mu^{\alpha\beta} = \beta^{\alpha\beta}$ as mentioned just before (33). We find

$$\delta\psi = \mathrm{i}\beta^{\alpha\beta}\sigma_{\alpha\beta}\psi\,. \tag{41}$$

The covariant derivative (28) of such fields, although resembling an ordinary H-covariant differential, is built with a non-linear connection defined on the whole G-algebra. Thus, a representation of the full Poincaré algebra is required in order to realize the non-linear connection (35) as

$$\rho(\Gamma_M) = -\mathrm{i}\,\Gamma^{\alpha\beta}\sigma_{\alpha\beta} - \mathrm{i}\,\vartheta^{\mu}\pi_{\mu}\,, \tag{42}$$

where $\pi_\mu = \rho(P_\mu)$ is the finite matrix representation of translational generators to be studied below. According to the general formula (28), the Poincaré covariant derivatives of Dirac fields read

$$D\psi = d\psi - \mathrm{i}\,(\Gamma^{\alpha\beta}\sigma_{\alpha\beta} + \vartheta^{\mu}\pi_{\mu})\psi\,, \tag{43}$$

transforming in analogy to (41) as

$$\delta D\psi = \mathrm{i}\,\beta^{\alpha\beta}\sigma_{\alpha\beta}D\psi\,. \tag{44}$$

Certainly, due to the particular non-linear Poincaré transformations (38) and (41), the contributions associated to the translational generators are not necessary to guarantee covariance of (43). Nevertheless, the general scheme requires these contributions to be present in the otherwise Lorentz covariant derivatives, as an unavoidable heritage of the gauged Poincaré group. So we need to know how the G generators not belonging to H act on the fields ψ of the representation space of H. In our case, this means that, besides (47), we have to look for the already mentioned representation of the translational generators in order to complete the finite matrix realization of the abstract Poincaré algebra (31) and (32).

3.2 Intrinsic translations of fermion fields

According to our conventions, the Dirac gamma matrices are defined so that their product reads

$$\gamma_\alpha\gamma_\beta = -o_{\alpha\beta}\,I - 4\mathrm{i}\,\sigma_{\alpha\beta}\,, \tag{45}$$

expressed in terms of the Minkowski metric

$$o_{\alpha\beta} := \mathrm{diag}\,(-+++) \tag{46}$$

and of the spinor generators

$$\sigma_{\alpha\beta} := \frac{\mathrm{i}}{8}\,[\,\gamma_\alpha\,,\gamma_\beta] \tag{47}$$

of the Lorentz group, being $\sigma_{\alpha\beta} = \rho(L_{\alpha\beta})$ the usual 4×4 matrix representation of the Lorentz algebra (31) acting on 4-dimensional Dirac bispinors ψ. Let us discuss how to extend the Lorentz algebra to the Poincaré algebra, the latter one constituting a subalgebra of the conformal algebra as shown in the appendix.

The possibility of constructing also intrinsic translational operators $\pi_\mu = \rho(P_\mu)$ from the gamma matrices rests on the fact that

$$[\,\sigma_{\alpha\beta}\,,\gamma_\mu] = \mathrm{i}\,o_{\mu[\alpha}\gamma_{\beta]}\,, \tag{48}$$

and on the properties of the γ_5 matrix, defined as

$$\gamma_5 := \mathrm{i}\,\gamma^0\gamma^1\gamma^2\gamma^3\,, \tag{49}$$

such that $\gamma_5^2 = I$ and satisfying the commutation relations

$$[\,\sigma_{\alpha\beta}\,,\gamma_5] = 0\,, \tag{50}$$

and the anticommutation relations

$$\{\,\gamma_\mu\,,\gamma_5\} = 0\,, \tag{51}$$

and

$$\{\,\sigma_{\alpha\beta}\,,\gamma_\mu\} = -\frac{1}{2}\,\eta_{\alpha\beta\mu}{}^{\nu}\gamma_\nu\gamma_5\,, \tag{52}$$

(where $\eta_{\alpha\beta\gamma\delta}$, with $\alpha\,,\beta\,,... = 0,...,3$, is defined so that $\eta_{0abc} = \epsilon_{abc}$, with $a,b,c = 1,2,3$). Making use of these elements, one finds operators

$$\pi_\mu \sim \gamma_\mu(\,1 + \lambda\,\gamma_5) \tag{53}$$

to exist, with $\lambda^2 = 1$, satisfying the commutation relations

$$[\,\sigma_{\alpha\beta}\,,\pi_\mu] = \mathrm{i}\,o_{\mu[\alpha}\pi_{\beta]}\,,\quad [\,\pi_\mu\,,\pi_\nu] = 0\,, \tag{54}$$

characteristic for translational generators; see (32). Notice that (54) do not completely determine π_μ. Actually, in (53) a global factor as much as the sign $\lambda\ (=\pm 1)$ remains unfixed. This fact reflects the existence of two inequivalent realizations of the full conformal algebra, of which the Poincaré algebra is a subalgebra. Invoking dimensionality consistence of the intrinsic linear momentum π_μ with the orbital linear momentum $-\mathrm{i}\partial_\mu$ we require the former, in natural units $\hbar = c = 1$, to have dimensions $[L]^{-1}$. Since the gamma matrices in (53) are dimensionless, we are enforced to introduce a dimensional constant, say $m \sim [L]^{-1}$. Let us also fix the undetermined sign in (53), see the appendix, and define

$$\pi_\mu := \frac{m}{4}\,\gamma_\mu(\,1 + \gamma_5)\,, \tag{55}$$

where the numerical factor is introduced for later convenience.

A remarkable feature of (55) is that $\pi_\mu\pi_\nu = 0$. The resulting anticommutation relations $\{\pi_\mu\,,\pi_\nu\} = 0$ are compatible with the finite matrix realization of the Poincaré algebra given by (47) and (55). Since the commutation relations alone are responsible for the transformations (33) of the coordinate-like parameters the matter fields depend on, they suffice to induce the change from $\psi(\sigma_\xi(x))$ into $\psi(\sigma_{\xi'}(x))$ where their gauge variation (14) is evaluated.

On the other hand, the usual Casimir characterization of mass still holds in our scheme despite the nilpotence of π_μ by considering the complete translational generators as consisting of the sum of an orbital plus an intrinsic contribution, namely $P_\mu = \mathrm{i}I\partial_\mu + \pi_\mu$. Observe that, in the limit of vanishing components of the Lorentz connections, the translational parameters ξ^μ become indistinguishable from Cartesian coordinates and the covariant derivative (43) reduces to the action of such a P_μ on fermions as $-\mathrm{i}\, d\xi^\mu P_\mu\psi$. Since π_μ is traceless and $\pi_\mu\pi_\nu = 0$, the Casimir relation $\mathrm{Tr}(P_\mu P^\mu) \sim m^2$ is valid for $m \neq 0$.

Our intrinsic translational generators (55) resemble the *momentum spin* introduced by Gürsey [33] in the context of the contraction of $O(3,2)$ to the Poincaré group [34]. Indeed, such a *momentum spin* is conceived as the intrinsic part of the pseudotranslational generators $\Pi_\mu := (1/R)\, L_{5\mu}$ whose commutation relations, in the limit $R \to \infty$, reproduce those of Poincaré translations.

4 Poincaré gauge invariant Dirac action

The discussion of previous section guarantees the translational contributions in (43) not only to make sense, but to be an essential part of (non-linear) Poincaré covariant derivatives. Thus we have all the elements needed to build the Dirac matter action in the presence of gravity, when the latter is described by (non-linear) PGT. Following the notation of [35], with $\gamma := \vartheta^\mu\gamma_\mu$, and ${}^*\gamma$ its Hodge dual, the Dirac Lagrange density 4-form – without explicit mass term – reads

$$L_D = \frac{\mathrm{i}}{2}\,\overline{\psi}\;{}^*\gamma \wedge D\psi + \text{h.c.}\,, \tag{56}$$

with the usual definition $\overline{\psi} := \psi^\dagger\gamma^0$, and h.c. standing for the Hermitian conjugate of the given term. Let us calculate the latter in order to make all our conventions explicit. From (45) we get $\gamma_0^2 = 1$. Provided

$$\gamma^0\gamma_\mu^\dagger\gamma^0 = \gamma_\mu\,, \tag{57}$$

as it is the case for instance for the Dirac representation of gamma matrices in terms of Pauli matrices as

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\,,\; \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}\,,\; \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\,, \tag{58}$$

we realize that

$$\left(\frac{\mathrm{i}}{2}\,\overline{\psi}\;{}^*\gamma \wedge D\psi\right)^\dagger = \frac{\mathrm{i}}{2}\,\overline{D\psi} \wedge {}^*\gamma\psi\,, \tag{59}$$

with $\overline{D\psi} := (D\psi)^\dagger\gamma^0$. Furthermore, (45) with (57) yields

$$\gamma^0\sigma_{\alpha\beta}^\dagger\gamma^0 = \sigma_{\alpha\beta}\,, \tag{60}$$

guaranteeing the invariance of (56) by enforcing $\overline{\psi}$ to transforms as

$$\delta\overline{\psi} = (\delta\psi)^\dagger\gamma^0 = -\mathrm{i}\overline{\psi}\,\beta^{\alpha\beta}\sigma_{\alpha\beta}\,, \tag{61}$$

and on the other hand from definition (49) with (57) we get

$$\gamma^0\gamma_5^\dagger\gamma^0 = -\gamma_5\,. \tag{62}$$

Applying (57) and (62) to (55), it follows that

$$\gamma^0\pi_\mu^\dagger\gamma^0 = \pi_\mu\,, \tag{63}$$

a result which was not a priori obvious. Taking (60) and (63) into account, from (43) we find

$$\overline{D\psi} := (D\psi)^\dagger\gamma^0 = d\overline{\psi} + \mathrm{i}\,\overline{\psi}(\Gamma^{\alpha\beta}\sigma_{\alpha\beta} + \vartheta^\mu\pi_\mu)\,, \tag{64}$$

transforming as

$$\delta\overline{D\psi} = -\mathrm{i}\,\overline{D\psi}\,\beta^{\alpha\beta}\sigma_{\alpha\beta}\,, \tag{65}$$

compare with (61). If desired, in order to take into account other forces besides gravitation, one can extend the gauge symmetry replacing the Poincaré group by the direct product of Poincaré times an internal group. To do so, one merely has to replace (43) by

$$D\psi = d\psi + \mathrm{i}\,(gA - \Gamma^{\alpha\beta}\sigma_{\alpha\beta} - \vartheta^\mu\pi_\mu)\psi \tag{66}$$

(and analogously (64)) without affecting what follows. In view of the previous results, the explicit form of (56) becomes

$$L_D = \frac{\mathrm{i}}{2}\,(\overline{\psi}\;{}^*\gamma \wedge D\psi + \overline{D\psi} \wedge {}^*\gamma\psi)\,. \tag{67}$$

Let us separate the translational parts, no more indispensable for the covariance of the covariant derivatives, from (43), (respectively (66)) as

$$D\psi =: \widetilde{D}\psi - \mathrm{i}\vartheta^\mu\pi_\mu\psi\,, \tag{68}$$

and analogously

$$\overline{D\psi} =: \overline{\widetilde{D}\psi} + \mathrm{i}\,\overline{\psi}\,\vartheta^\mu\pi_\mu\,, \tag{69}$$

see (64), where the tildes denote the translations-independent pieces. Replacing (68) and (69) in (67), the Lagrange density transforms into

$$L_D = \frac{\mathrm{i}}{2}\,(\overline{\psi}\;{}^*\gamma \wedge \widetilde{D}\psi + \overline{\widetilde{D}\psi} \wedge {}^*\gamma\psi) + {}^*m\,\overline{\psi}\psi\,, \tag{70}$$

where we made use of the fact that $\vartheta^\alpha \wedge {}^*\vartheta_\beta = \delta^\alpha_\beta\,\eta$, with $\eta = {}^*1$ as the 4-dimensional volume element, so that

$${}^*\gamma \wedge \vartheta^\mu\pi_\mu = -\eta\,\gamma^\mu\pi_\mu = {}^*m\,(1+\gamma_5)\,, \tag{71}$$

and

$$-\vartheta^\mu\pi_\mu \wedge {}^*\gamma = -\eta\,\pi_\mu\gamma^\mu = {}^*m\,(1-\gamma_5)\,. \tag{72}$$

Although γ_5 is necessary to guarantee the commutation relations (54) to hold, both contributions (71) and (72) are combined in the action in such a way that γ_5 cancels out. So the matter Lagrange density (70) merely retains a mass term, which is unavoidable since it derives from the translational contribution to the Poincaré connection (42). Accordingly, either one of the projections ψ_L or ψ_R is lacking (in which case $\overline{\psi}\psi = 0$), or otherwise the field ψ is necessarily massive.

5 Conclusions

Independently from other possible origins of fermion masses, a gravitational background mass contribution is predicted by PGT when treated as a non-linear local realization of the Poincaré group. Provided both left and right projections of Dirac fields are simultaneously present, (70) prevents massless Dirac fields from existing. The irremovable fermion masses are a consequence of gravitational interaction (in particular of the underlying translational group) in the context of PGT as the fundamental theory of gravity.

As a phenomenological consequence, when considered together with the standard model, PGT gives rise to a background contribution of gravitational origin to the masses of all fermions: in particular to the quark mass parameters of the QCD sector of the Lagrangian, as much as to the neutrino masses. Neutrinos are thus predicted by PGT to be massive. Certainly, our approach does not determine the value of the universal translational mass parameter m. However, from the observed masses of neutrinos it is clear that m (the same for all fermions) has to be very small, so that, accordingly, its contribution to the observable hadron masses is expected to be quite limited.

Matter currents corresponding to the Poincaré symmetry are the spin current $\tau_{\alpha\beta} := \partial L_D / \partial \Gamma^{\alpha\beta}$ and the energy-momentum 3-form $\Sigma_\mu := \partial L_D / \partial \overset{(T)}{\Gamma^\mu} = \partial L_D / \partial \vartheta^\mu$. The former is found to be $\tau_{\alpha\beta} = -\frac{1}{4}\,\overline{\psi}\,\vartheta_\alpha \wedge \vartheta_\beta \wedge \gamma\gamma_5\psi$ as it is well known. Its coupling term to the Lorentz connection $\Gamma^{\alpha\beta}$ falls off from the action in the limit of absence of gravity (that is for $\Gamma^{\alpha\beta} = 0$, $\overset{(T)}{\Gamma^\mu} = 0$). Instead, the mass term does not cancel out in this limit. The reason is that, according to the non-linear approach to PGT, the tetrads have the structure (36), not vanishing for zero linear connections. Actually, ordinary Minkowskian flat spacetime may be regarded as the residual structure left by non-linear PGT in the absence of the gravitational force carried by spin connections, that is in the limit where the components of the latter ones are chosen to vanish. The tetrads are in this case $\vartheta^\mu = d\xi^\mu$, so that the mass term associated to them still remains in the action despite translational linear connections are switched out.

Appendix A: The O(2,4) and the Poincaré algebra

The Poincaré algebra is a subalgebra of the conformal algebra [36] to be examined here. Consider the $O(2,4)$ generators $L_{AB} = -L_{BA}$, $A, B, ... = 0, ..., 3, 5, 6$, satisfying the commutation relations

$$[\, L_{AB}\,, L_{MN}] = -\mathrm{i}\,(g_{A[M}L_{N]B} - g_{B[M}L_{N]A})\,, \qquad \text{(A.1)}$$

where the 6-dimensional metric tensor is taken to be

$$g_{AB} = \mathrm{diag}\,(-+++,+-)\,. \qquad \text{(A.2)}$$

In order to relate (A.1) to the ordinary form of the conformal commutation relations, let us decompose (A.2) into the Minkowski metric

$$g_{\alpha\beta} = o_{\alpha\beta} := \mathrm{diag}\,(-+++)\,, \qquad \text{(A.3)}$$

where $\alpha, \beta = 0, ..., 3$, plus

$$g_{55} = 1\,, \quad g_{66} = -1\,, \qquad \text{(A.4)}$$

and define the translational generators

$$P_\mu := L_{\mu 5} + L_{\mu 6}\,, \qquad \text{(A.5)}$$

the special conformal generators

$$K_\mu := L_{\mu 5} - L_{\mu 6}\,, \qquad \text{(A.6)}$$

and the dilatational generators

$$D := -2\,L_{56}\,. \qquad \text{(A.7)}$$

In terms of $L_{\alpha\beta}$, (A.5), (A.6) and (A.7), the commutation relations (A.1) give rise to the conformal algebra

$$[\, L_{\alpha\beta}\,, L_{\mu\nu}] = -\mathrm{i}\,(o_{\alpha[\mu}L_{\nu]\beta} - o_{\beta[\mu}L_{\nu]\alpha})\,, \qquad \text{(A.8)}$$
$$[\, L_{\alpha\beta}\,, P_\mu] = \mathrm{i}\,o_{\mu[\alpha}P_{\beta]}\,, \qquad \text{(A.9)}$$
$$[\, L_{\alpha\beta}\,, K_\mu] = \mathrm{i}\,o_{\mu[\alpha}K_{\beta]}\,, \qquad \text{(A.10)}$$
$$[\, P_\mu\,, K_\nu] = \mathrm{i}\left(L_{\mu\nu} + \frac{1}{2}\,o_{\mu\nu}D\right)\,, \qquad \text{(A.11)}$$
$$[\, D\,, P_\mu] = -\mathrm{i}\,P_\mu\,, \qquad \text{(A.12)}$$
$$[\, D\,, K_\mu] = \mathrm{i}\,K_\mu\,, \qquad \text{(A.13)}$$
$$[\, P_\mu\,, P_\nu] = [\, K_\mu\,, K_\nu] = [\, D\,, L_{\mu\nu}] = [\, D\,, D] = 0\,. \qquad \text{(A.14)}$$

As pointed out in [36], all finite-dimensional representations of the $O(2,4)$-algebra can be obtained by reducing out tensor products of two inequivalent fundamental 4-dimensional representations (corresponding respectively to the choices $\lambda = \pm 1$ in what follows) built from the gamma matrices as

$$\rho(L_{\alpha\beta}) = \sigma_{\alpha\beta} := \frac{\mathrm{i}}{8}\,[\,\gamma_\alpha\,, \gamma_\beta]\,, \qquad \text{(A.15)}$$
$$\rho(L_{\mu 5}) = \frac{1}{2}\,(\pi_\mu + \kappa_\mu) = \lambda \frac{m}{4}\,\gamma_\mu\gamma_5\,, \qquad \text{(A.16)}$$
$$\rho(L_{\mu 6}) = \frac{1}{2}\,(\pi_\mu - \kappa_\mu) = \frac{m}{4}\,\gamma_\mu\,, \qquad \text{(A.17)}$$
$$\rho(L_{56}) = -\frac{1}{2}\,\Delta = -\lambda\,\frac{\mathrm{i}}{4}\,\gamma_5\,. \qquad \text{(A.18)}$$

Obviously, as read out from (A.16), (A.17) and (A.18), the corresponding fundamental inequivalent representations of (A.5), (A.6) and (A.7) read

$$\pi_\mu := \frac{m}{4}\,\gamma_\mu(\,1 + \lambda\gamma_5)\,, \qquad \text{(A.19)}$$
$$\kappa_\mu := -\frac{m}{4}\,\gamma_\mu(\,1 - \lambda\gamma_5)\,, \qquad \text{(A.20)}$$

and

$$\Delta := \lambda \frac{i}{2} \gamma_5 , \qquad (A.21)$$

where the role of π_μ and κ_μ is interchangeable by fixing λ to be either ± 1, and by accordingly change the sign of (A.21). The Poincaré algebra considered in the main text is the subalgebra of the conformal algebra consisting of the spin and translational generators only, having fixed $\lambda = 1$.

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1 Utiyama:1956sy 2 Sciama1 3 Sciama2 4 Kibble:1961ba 5 Hayashi:1980wj,6 Lord:1987uq 7 Lord:1988nd 8 Hehl:1995ue, 9 Gronwald:1995em 10 Ivanenko:1983vf 11 Sardanashvily:2002mi 12 Hehl:1974cn 13 Hehl:1976kj 14 Julve:1994bh 15 Lopez-Pinto:1995qb 16 Lopez-Pinto:1997aw 17 Tresguerres:2000qn 18 Tresguerres:2002uh 19 Sardanashvily:1992fq 20 Sardanashvily:1994 21 Sardanashvily:1994fg 22 Sardanashvily:1995ew 23 Kobayashi:1963 24 Cho:1978ss 25 Coleman:1969sm 26 Callan:1969sn 27 Salam:1969rq 28 Isham:1971dv 29 Borisov74 30 Stelle:1980aj 31 Tseytlin:1982nu 32 Gursey:1964 33 Inonu:1953sp 34 Hehl:1990yq

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